

Asymptotic for a second order evolution equation with convex potential and vanishing  
damping term

Ramzi MAY

Department of Mathematics and Statistics

College of Sciences

King Faisal University

P.O. 400 Al Ahsaa 31982, Kingdom of Saudi Arabia

E-mail: rmay@kfu.edu.sa

**Abstract:** In this short note, we recover by a different method the new result due to Attouch, Chbani, Peyrouquet and Redont concerning the weak convergence as  $t \rightarrow +\infty$  of solutions  $x(t)$  to the second order differential equation

$$x''(t) + \frac{K}{t}x'(t) + \nabla\Phi(x(t)) = 0,$$

where  $K > 3$  and  $\Phi$  is a smooth convex function defined on an Hilbert Space  $\mathcal{H}$ . Moreover, we improve their result on the rate of convergence of  $\Phi(x(t)) - \min \Phi$ .

**keywords:** dynamical systems, asymptotically small dissipation, asymptotic behavior, energy function, convex function, convex optimization.

## 1. INTRODUCTION AND STATEMENT OF THE RESULT

Let  $\mathcal{H}$  be a real Hilbert space with inner product and norm respectively denoted by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ . In a very recent work [1], Attouch, Chbani, Peypouquet and Redont have considered the following second order differential equation:

$$(1.1) \quad x''(t) + \gamma(t)x'(t) + \nabla\Phi(x(t)) = 0,$$

where  $\gamma(t) = \frac{K}{t}$  with  $K$  is a non negative constant and  $\Phi : \mathcal{H} \rightarrow \mathbb{R}$  is a convex continuously differentiable function. By developing a method due to Su, Boyd, and Condes [4], they proved the following result:

**Theorem 1.1** (Attouch, Chbani, Peypouquet, and Redont). *Assume that  $K > 3$  and the set  $\arg \min \Phi \equiv \{x \in \mathcal{H} : \Phi(x) \leq \Phi(y) \forall y \in \mathcal{H}\}$  is nonempty. Let  $x : [t_0, +\infty[ \rightarrow \mathcal{H}$  be a solution to (1.1). Then  $x(t)$  converges weakly in  $\mathcal{H}$  as  $t \rightarrow +\infty$  to some element of  $\arg \min \Phi$ . Moreover the energy function*

$$(1.2) \quad W(t) \equiv \frac{1}{2} \|x'(t)\|^2 + \Phi(x(t)) - \min \Phi$$

*satisfies  $W(t) = O(t^{-2})$  as  $t \rightarrow +\infty$ .*

In this note, we establish, by using a different method, a slightly improved version of the previous theorem. Precisely, we prove the following result.

**Theorem 1.2.** *Assume that  $K > 3$  and  $\arg \min \Phi \neq \emptyset$ . Let  $x : [t_0, +\infty[ \rightarrow \mathcal{H}$  be a solution to (1.1). Then  $x(t)$  converges weakly in  $\mathcal{H}$  as  $t \rightarrow +\infty$  to some element of  $\arg \min \Phi$ . Moreover  $W(t) = o(t^{-2})$  as  $t \rightarrow +\infty$ .*

**Remark 1.1.** *In [3], we studied the asymptotic behavior as  $t \rightarrow +\infty$  of solution to equation (1.1) when the damping term  $\gamma(t)$  behaves, for  $t$  large enough, like  $\frac{K}{t^\alpha}$  with  $K > 0$  and  $\alpha \in [0, 1[$ . We proved that if  $\arg \min \Phi \neq \emptyset$  then every solution to (1.1) converges weakly in  $\mathcal{H}$  to some element of  $\arg \min \Phi$ . Hence, Theorem 1.1 and Theorem 1.2 extend this result to the limit case corresponding to  $\alpha = 1$ .*

## 2. PROOF OF THEOREM 1.2

We will prove Theorem 1.2 in a more general setting. Indeed, we will assume that the damping term  $\gamma$  in Equation (1.1) is a real function defined on  $[t_0, +\infty[$  which belongs to the class  $W_{loc}^{1,1}([t_0, +\infty[, \mathbb{R})$  and satisfies:

$$(2.1) \quad \text{There exists } K > 3 \text{ such that } \gamma(t) \geq \frac{K}{t} \quad \forall t \geq t_0,$$

and

$$(2.2) \quad \int_{t_0}^{+\infty} [(t\gamma(t))'_+] dt < +\infty,$$

where  $[(t\gamma(t))'_+] \equiv \max\{(t\gamma(t))', 0\}$  is the positive part of  $(t\gamma(t))'$ .

A typical examples of functions  $\gamma$  satisfying (2.1) and (2.2) are  $\gamma(t) = \frac{K}{a+t}$  with  $a \in \mathbb{R}$  and  $K > 3$ .

*Proof of Theorem 1.2.* We will use a modified version of a method introduced by Cabot et Frankel in [2] and recently developed in [3].

Let  $x^* \in \arg \min \Phi$  and define the function  $h : [t_0, +\infty[ \rightarrow \mathbb{R}^+$  by  $h(t) = \frac{1}{2} \|x(t) - x^*\|^2$ . By differentiating, we have

$$\begin{aligned} h'(t) &= \langle x'(t), x(t) - x^* \rangle, \\ h''(t) &= \|x'(t)\|^2 + \langle x''(t), x(t) - x^* \rangle. \end{aligned}$$

Combining these last equalities and using Equation (1.1), we get

$$(2.3) \quad h''(t) + \gamma(t)h'(t) = \|x'(t)\|^2 + \langle \nabla \Phi(x(t)), x^* - x(t) \rangle.$$

Using now the convexity inequality

$$(2.4) \quad \Phi(x^*) \geq \Phi(x) + \langle \nabla \Phi(x), x^* - x \rangle,$$

and the definition (1.2) of the energy function  $W$ , we obtain

$$(2.5) \quad W(t) \leq \frac{3}{2} \|x'(t)\|^2 - h''(t) - \gamma(t)h'(t).$$

On the other hand, in view of (1.1),

$$\begin{aligned} W'(t) &= \langle x'(t), x(t) \rangle + \langle \nabla \Phi(x(t)), x'(t) \rangle \\ &= -\gamma(t) \|x'(t)\|^2. \end{aligned}$$

Hence

$$(2.6) \quad (t^2 W(t))' = 2tW(t) - t^2 \gamma(t) \|x'(t)\|^2.$$

Using now assumption (2.1), we get

$$\begin{aligned} \frac{3}{2} t \|x'\|^2 &\leq \frac{3}{2K} t^2 \gamma(t) \|x'(t)\|^2 \\ (2.7) \quad &= \frac{3}{K} t W(t) - \frac{3}{2K} (t^2 W(t))'. \end{aligned}$$

Multiplying (2.5) by  $t$  and using Inequality (2.7), we obtain

$$(1 - \frac{3}{K}) t W(t) + \frac{3}{2K} (t^2 W(t))' \leq -t h''(t) - t \gamma(t) h'(t).$$

Integrating this last inequality on  $[t_0, t]$ , we get after simplification

$$\begin{aligned} (1 - \frac{3}{K}) \int_{t_0}^t s W(s) ds + \frac{3}{2K} (t^2 W(t)) &\leq C_0 - t h'(t) + (1 - t \gamma(t)) h(t) \\ (2.8) \quad &+ \int_{t_0}^t (s \gamma(s))' h(s) ds, \end{aligned}$$

where  $C_0 = \frac{3}{2K} (t_0^2 W(t_0)) + t_0 h'(t_0) - h(t_0)$ .

Let  $\varepsilon > 0$  such that  $K > 3 + 3\varepsilon$ . By using (2.1), we obtain from the inequality (2.8)

$$\begin{aligned} (1 - \frac{3}{K}) \int_{t_0}^t s W(s) ds + \frac{3}{2K} (t^2 W(t)) + \varepsilon h(t) &\leq C_0 - t h'(t) - (K - 1 - \varepsilon) h(t) \\ &+ \int_{t_0}^t [(s \gamma(s))']_+ h(s) ds. \end{aligned}$$

Using now the fact that

$$\begin{aligned} t |h'(t)| &\leq t \|x'(t)\| \|x(t) - x^*\| \\ &\leq 2 \sqrt{t^2 W(t)} \sqrt{h(t)}, \end{aligned}$$

and applying the elementary inequality

$$\forall a > 0 \forall b, x \in \mathbb{R}, -ax^2 + bx \leq \frac{b^2}{4a}$$

with  $x = \sqrt{h(t)}$ , we get

$$(2.9) \quad A \int_{t_0}^t sW(s)ds + Bt^2W(t) + \varepsilon h(t) \leq C_0 + \int_{t_0}^t [(s\gamma(s))'_+] h(s)ds$$

where  $A = 1 - \frac{3}{K}$  and  $B = \frac{3}{2K} - \frac{1}{K-1-\varepsilon}$ .

Since  $K > 3 + 3\varepsilon$ , the constants  $A$  and  $B$  are positive, then

$$\varepsilon h(t) \leq C_0 + \int_{t_0}^t [(s\gamma(s))'_+] h(s)ds.$$

Hence, by using Gronwall's inequality and the assumption (2.2), we deduce that the function  $h$  is bounded, more precisely, we get

$$\sup_{t \geq t_0} h(t) \leq \frac{C_0}{\varepsilon} \exp\left(\frac{1}{\varepsilon} \int_{t_0}^{+\infty} [(s\gamma(s))'_+] ds\right).$$

Therefore, we infer from (2.9) that

$$(2.10) \quad \sup_{t \geq t_0} t^2 W(t) < +\infty,$$

$$(2.11) \quad \int_{t_0}^{+\infty} sW(s)ds < +\infty.$$

Combining (2.6) and (2.11) yields that the positive part  $[(t^2W(t))'_+]_+$  of  $(t^2W(t))'$  belongs to  $L^1([t_0, +\infty[, \mathbb{R})$ , hence  $m := \lim_{t \rightarrow +\infty} t^2W(t)$  exists. This limit  $m$  must be equal to 0, since otherwise  $tW(t) \simeq \frac{m}{t}$  as  $t \rightarrow +\infty$  which contradicts (2.11). It remains to prove the weak convergence of  $x(t)$  as  $t \rightarrow +\infty$ . Let us notice that (2.10) implies that  $\Phi(x(t)) \rightarrow \min \Phi$  as  $t \rightarrow +\infty$ . Hence by using the weak lower semi-continuity of the function  $\Phi$ , we deduce that if  $x(t_n) \rightharpoonup \bar{x}$  weakly in  $\mathcal{H}$  with  $t_n \rightarrow +\infty$  then  $\Phi(\bar{x}) \leq \min \Phi$  which is equivalent to  $\bar{x} \in \arg \min \Phi$ . On the other hand, from the convex inequality (2.4) we deduce that  $\langle \nabla \Phi(x), x^* - x \rangle \leq 0$  for every  $x \in \mathcal{H}$ . Then Equation (2.3) implies

$$h''(t) + \gamma(t)h'(t) \leq \|x'(t)\|^2.$$

Multiply this last equation by  $e^{\Gamma(t, t_0)}$ , where  $\Gamma(t, s) = \int_s^t \gamma(\tau) d\tau$ , and integrate between  $t_0$  and  $t$ , we obtain

$$(2.12) \quad h'(t) \leq e^{-\Gamma(t, t_0)} h'(t_0) + \int_{t_0}^t e^{-\Gamma(t, \tau)} \|x'(\tau)\|^2 d\tau.$$

In view of the assumption (2.1), a simple calculation gives

$$\forall s \geq t_0, \int_s^{+\infty} e^{-\Gamma(t,s)} dt \leq \frac{s}{K-1}.$$

Hence by using (2.12) and Fubini Theorem, we get

$$\int_{t_0}^{+\infty} [h'(t)]_+ dt \leq \frac{t_0 |h'(t_0)|}{K-1} + \frac{1}{K-1} \int_{t_0}^{+\infty} \tau \|x'(\tau)\|^2 d\tau.$$

Thanks to (2.11), the right hand side of the last inequality is finite, thus  $\int_{t_0}^{+\infty} [h'(t)]_+ dt < +\infty$  which implies that  $\lim_{t \rightarrow +\infty} h(t)$  exists. Hence, for every  $x^* \in \arg \min \Phi$ , the limit of  $\|x(t) - x^*\|$  as  $t \rightarrow +\infty$  exists. Therefore, Opial's lemma [5], which we recall below, guaranties the required weak convergence of  $x(t)$  in  $\mathcal{H}$  to some element of  $\arg \min \Phi$ .  $\square$

**Lemma 2.1** (Opial's lemma). *Let  $x : [t_0, +\infty[ \rightarrow \mathcal{H}$ . Assume that there exists a nonempty subset  $S$  of  $\mathcal{H}$  such that:*

- i) *If  $t_n \rightarrow +\infty$  and  $x(t_n) \rightharpoonup x$  weakly in  $\mathcal{H}$ , then  $x \in S$ .*
- ii) *For every  $z \in S$ ,  $\lim_{t \rightarrow +\infty} \|x(t) - z\|$  exists.*

*Then there exists  $z_\infty \in S$  such that  $x(t) \rightharpoonup z_\infty$  weakly in  $\mathcal{H}$  as  $t \rightarrow +\infty$ .*

**Conclusion:** In this paper, we have proved that if the damping term  $\gamma(t)$  behaves at infinity like  $\frac{K}{t}$  with  $K > 3$ , then every solution  $x(t)$  of the equation (1.1) converges weakly as  $t \rightarrow +\infty$  to a minimizer of  $\Phi$  and the energy function  $W(t)$  is  $\circ(t^{-2})$ . However, two important questions remain open. The first one is on the behavior of the solution  $x(t)$  in the limit case  $K = 3$  and the second one is about the effect of the constant  $K$  on the convergence rate of the associated energy function  $W(t)$ .

**Acknowledgement:** The author wish to thank Prof. Adel Trabelsi for his comments which were very useful to improve the representation of the paper.

## REFERENCES

- [1] Attouch H, Chbani Z, Peypouquet J, Redont P. Fast convergence of inertial dynamics and algorithms with asymptotic vanishing viscosity. Math Program Ser B 2016; 1-53.
- [2] Cabot A, Frankel P. Asymptotics for some semilinear hyperbolic equations with non-autonomous damping. J Differ Equations 2012; 252: 294-322.
- [3] May R. Long time behavior for a semilinear hyperbolic equation with asymptotically vanishing damping term and convex potential. J Math Anal Appl 2015; 430: 410-416.
- [4] Su W, Boyd S, Candes E J. A differential equation for modeling Nesterov's accelerated gradient method: Theory and Insights. Neural Information Proceeding Systems (NIPS). 2014.
- [5] Opial Z. Weak convergence of the sequence of successive approximation for nonexpansive mapping. Bull Amer Math Soc 1967; 73: 591-597.